On the Relationship between Arithmetic and Geometric Returns

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Abstract

In this short article, I prove that the Arithmetic mean return is always higher than the Geometric mean return.

1 Introduction

Arithmetic mean return is computed as

$$AM = \frac{1}{N}\sum_{k=1}^{N} r_k = \frac{1}{N}\sum_{k=1}^{N}(r_k + 1) - 1$$

Geometric mean return is computed as

$$GM = \prod_{k=1}^{N}(1 + r_k)^{\frac{1}{N}} - 1$$

2 Proof using Jensen's Inequality

To prove that $AM \geq GM$, we need to prove

$$\frac{1}{N}\sum_{k=1}^{N}(r_k + 1) - 1 \geq \prod_{k=1}^{N}(1 + r_k)^{\frac{1}{N}} - 1$$

Or equivalently,

$$\frac{1}{N}\sum_{k=1}^{N}(r_k + 1) \geq \prod_{k=1}^{N}(1 + r_k)^{\frac{1}{N}}$$

Recall that $f = \ln(x)$ is strictly concave. According to Jensen's Inequality for concave functions, we have

$$f(\sum_{k=1}^{N} a_k x_k) \geq \sum_{k=1}^{N} a_k f(x_k)$$

Where x_k is strictly non-negative. Let $a_k = \frac{1}{N}$, $x_k = r_k + 1$, this inequality implies

$$\begin{aligned} \ln(\sum_{k=1}^{N} \frac{1}{N}(r_k + 1)) &\geq \sum_{k=1}^{N} \frac{1}{N} \ln(r_k + 1) \\ &= \ln(\prod_{k=1}^{N} (r_k + 1)^{\frac{1}{N}}) \end{aligned}$$

Taking exponential on both sides yields,

$$\sum_{k=1}^{N} \frac{1}{N}(r_k + 1) \geq \prod_{k=1}^{N} (r_k + 1)^{\frac{1}{N}}$$

Which implies

$$\frac{1}{N} \sum_{k=1}^{N} r_k \geq \prod_{k=1}^{N} (r_k + 1)^{\frac{1}{N}} - 1$$

Or equivalently,

$$AM \geq GM$$